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INFINITE DISTANCE TRANSITIVE GRAPHS OF FINITE VALENCY

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We determine all infinite distance transitive graphs of finite valency, thereby proving a conjecture of C. D. Godsil. The proof makes heavy use of a theorem of M. J. Dunwoody concerning cuts of infinite graphs. In section 1 there is a rough analysis of the appearance of such graphs, and in section 2 we state and apply Dunwoody's theorem. The proof is completed in section 3.

1. Applications of the intersection array

We consider simple undirected graphs Γ, with vertex set $V\Gamma$. If vertices x, y are adjacent, we write $x \sim y$. For $x, y \in V\Gamma$, the distance $d(x, y)$ is defined in the usual way: We say that Γ is *distance transitive* if it satisfies the following condition: for any $x, y, x', y' \in V\Gamma$ with $d(x, y) = d(x', y')$, there is $g \in \operatorname{Aut}\Gamma$ with $xg = x'$ and $yg = y'$. When Γ is distance transitive we can define the *intersection numbers* c_i, a_i and b_i of [1]. That is, if $d(x, y) = i$, then

$$c_i = |\{z \in V\Gamma :\ d(x, z) = 1,\ d(y, z) = i-1\}|$$

$$a_i = |\{z \in V\Gamma :\ d(x, z) = 1,\ d(y, z) = i\}|$$

$$b_i = |\{z \in V\Gamma :\ d(x, z) = 1,\ d(y, z) = i+1\}|.$$

Finally, by a *cycle* of length n we mean a set $\{x_1, \dots, x_n\}$ of vertices carrying edges $x_i x_{i+1}$ (for $i = 1, \dots, n-1$) and $x_n x_1$. A cycle C is said to be *non-degenerate* if, whenever $x, y \in C$, there is a path in C of length $d(x, y)$ between x and y.

1.1. Definition. We say that an infinite graph Γ is *standard* if (i) there are no non-degenerate cycles in Γ of length greater than 3
(ii) there are finite parameters s, t such that the vertices adjacent to any vertex carry a subgraph consisting of s disjoint copies of the t-vertex complete graph.

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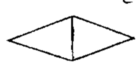
Any standard graph arises in the following way: let T be a tree in which the vertices of the bipartite blocks have valencies s, $t+1$ respectively. Then there is a standard graph whose vertices are the s-valent vertices of T, two vertices being joined if they lie at distance 2 in T. Informally, a standard graph is a treelike form of complete graphs. Clearly any standard graph is distance transitive.

1.2 Theorem. *If Γ is an infinite distance transitive graph of finite valency, then it is standard.*

The original and equivalent formulation was: if Γ satisfies the hypotheses of 1.2, then it is not 2-connected. In the rest of this paper, Γ is an infinite distance transitive graph of finite valency. To prove the theorem, it will suffice to show that Γ has no non-degenerate cycles of length greater than 3, and no subgraphs of the form



. In this section we show, by considering the intersection numbers, that Γ has a roughly treelike structure.

1.3 Lemma. *There is a positive integer d such that, for all $i>d$, $c_i=c_{i+1}$, $a_i=a_{i+1}$ and $b_i=b_{i+1}$.*

Proof. It is shown in Biggs [1] that $c_i \leqq c_{i+1}$ and $b_i \geqq b_{i+1}$ for any $i \in \mathbf{N}$. Further, $c_i+a_i+b_i$ is the valency of Γ, so is constant. The result follows immediately. ∎

Now fix $x \in V\Gamma$, and for each $y \in V\Gamma$, let

$$S_c(y) = \{w \in V\Gamma:\ w \sim x,\ d(y, w) = d(y, x)-1\}$$

$$S_a(y) = \{w \in V\Gamma:\ w \sim x,\ d(y, w) = d(y, x)\}$$

$$S_b(y) = \{w \in V\Gamma:\ w \sim x,\ d(y, w) = d(y, x)+1\}.$$

Let $\Omega_x=\{y \in V\Gamma:\ d(x, y) \geqq d+1\}$, and define three relations $\equiv_c$, $\equiv_a$ and $\equiv_b$ on $V\Gamma$ as follows: for $y, z \in V\Gamma$, put

$$y \equiv_c z \quad \text{whenever} \quad S_c(y) = S_c(z)$$

$$y \equiv_a z \quad \text{whenever} \quad S_a(y) = S_a(z)$$

$$y \equiv_b z \quad \text{whenever} \quad S_b(y) = S_b(z).$$

Clearly $\equiv_c$, $\equiv_a$ and $\equiv_b$ are equivalence relations on $V\Gamma$.

1.4 Lemma. *If y, z lie in the same connected component of Ω_x, then $y \equiv_c z$, $y \equiv_a z$ and $y \equiv_b z$.*

Proof. We first show that $y \equiv_c z$. It is enough to show that if $y, z \in \Omega_x$ and $y \sim z$, then $y \equiv_c z$. There are three cases. Let $d(x, y)=i$

(i) $d(y, x)=d(z, x)-1$. Now $S_c(y) \subseteq S_c(z)$; but $|S_c(y)|=|S_c(z)|$, so $S_c(y)=S_c(z)$.

(ii) $d(y, x)=d(z, x)+1$. Then $S_c(y) \supseteq S_c(z)$, again giving the result.

(iii) $d(y, x)=d(z, x)$. Choose $u \in S_c(y)$. First, $\{w \in V\Gamma:\ w \sim y,\ d(w, u) \leqq i-1\} \subseteq$

$\subseteq \{w\in V\Gamma\colon w\sim y,\ d(w,x)\leqq i\}$. These two sets have the same cardinalities (as $c_{i-1}+a_{i-1}=c_i+a_i$), so they are equal. Second, $\{w\in V\Gamma\colon w\sim y, d(w,u)<i-1\}=\{w\in V\Gamma\colon w\sim y, d(w,x)<i\}$. Hence, $\{w\in V\Gamma\colon w\sim y, d(w,u)=i-1\}=\{w\in V\Gamma\colon w\sim y, d(w,x)=i\}$. As z lies in the second of these sets, it lies in the first, so $y\equiv_c z$.

Applying this result, it is easy to show that if $y, z\in\Omega_x$ and $y\sim z$, then $y\equiv_a z$ and $y\equiv_b z$. ▮

For the rest of the paper, the integer d is that mentioned in 1.3. For any vertex x and positive integer h, we put

$$\Gamma_h(x) = \{y\in V\Gamma\colon\ d(x,y) = h\}.$$

1.5 Lemma. (a) *Suppose vertices x and y are adjacent in Γ, and C is a connected component of $\Omega_x\cap\Omega_y$. Then for $h>d+1$, the set $\Gamma_h(x)\cap C$ is one of $\Gamma_{h-1}(y)\cap C$, $\Gamma_h(y)\cap C$, or $\Gamma_{h+1}(y)\cap C$.*

(b) *Let $x\sim x_1\sim\ \dots\ \sim x_n\sim y$ be a path from x to y, and let C be a connected component of $\Omega_x\cap\Omega_y\cap\left(\bigcap_{i=1}^{n}\Omega_{x_i}\right)$. Then the sets $\{\{\Gamma_h(x)\cap C\}\colon h\geqq d+2\}$ and $\{\{\Gamma_h(y)\cap C\}\colon h\geqq d+2\}$ are equal.*

Proof. (a) Direct from 1.4.

(b) This follows immediately from (a). ▮

1.6 Definitions. A *geodesic* between any two vertices of Γ is any shortest path between those vertices. As Γ has infinite diameter, there are arbitrarily large subgraphs of Γ of the form

o—o—o—...—o—o—o

where the subgraph contains a Γ-geodesic between any two of its points. We call such subgraphs *lines*. It is clearly possible, using the axiom of choice, to extend lines infinitely in one or two directions, so we talk also of infinite one-way or two-way lines. Theorem 1.2 does not depend upon the axiom of choice, since in the proof it is always sufficient to consider large finite lines, though for ease of exposition we use infinite lines. The set Π is said to be the branch on a line ℓ, if Π is the subgraph consisting of all the vertices lying on geodesics between points of ℓ. In general, if $x, y, z\in V\Gamma$ and $d(x,y)$, $d(x,z)$, $d(y,z)>d+1$, we say that y is on the (x,z) branch if some geodesic from x to z passes through y, that is if $d(x,y)+d(y,z)=d(x,z)$. These definitions are justified by the next lemma.

1.7 Lemma. (a) *If $u_1, \dots, u_n$ lie on the (x,y) branch, with $d(x,y)$, $d(x,u_i)$, $d(y,u_i)$, $d(u_i,u_j)>d+1$ for distinct $i,j\in\{1,\dots,n\}$, then there is some geodesic from x to y which passes through each of $u_1,\dots,u_n$.*

(b) *If z is on the (x,y) branch, and x, y are on the (u,v) branch, the five points being pairwise at distance at least $d+1$ from each other, then z lies on the (u,v) branch.*

Proof. (a) By an inductive argument, it is sufficient to consider the case $n=2$. Suppose that $n=2$, and $d(x,u_1)<d(x,u_2)$. Since u_2 and y lie in the same component of $\Omega_x\cap\Omega_{u_1}$, it follows from 1.5(b) that $d(u_1,y)-d(u_1,u_2)=d(x,y)-d(x,u_2)=d(u_2,y)$. Hence, $d(x,y)=d(x,u_1)+d(u_1,y)=d(x,u_1)+d(u_1,u_2)+d(u_2,y)$.

(b) This follows directly from (a). ∎

Informally, Γ has the appearance of a tree with branches of unknown thickness. We say that T is a cross-section of a branch Π on Γ if T is maximal with respect to the property that $T \subseteq \Gamma_h(u) \cap \Gamma_{h+1}(v) \cap \Pi$ for some integer $h > d+1$ and some $u, v \in \ell$ with $u \sim v$. Any Π cross-section T contains a unique point of ℓ and $\Pi \setminus T$ has two components. By distance transitivity applied to the lines, cross-sections have constant size throughout Γ. The branch Π is the disjoint union of Π cross-sections.

2. Cuts of Γ_Δ

We state Theorem 1.1 of Dunwoody [2], and first introduce the terminology. If Δ is any locally finite graph, the number $e(\Delta)$ of ends of Δ is the supremum of the number of infinite connected components which can be obtained by removing finitely many edges of Δ. For any set $C \subseteq V\Delta$, let

$$\delta C = \{e \in E\Delta : \text{just one vertex of } e \text{ lies in } C\},$$

where $E\Delta$ is the edge set of Δ. We say that C is a cut if $C \subseteq V\Delta$ and δC is finite. A cut C is said to be non-trivial if both C and C^* (where $C^* = V\Delta \setminus C$) are infinite.

2.1 Theorem (Dunwoody). *Let Δ be a graph with more than one end. Then there is a non-trivial cut $C \subseteq V\Delta$ such that for any $g \in \mathrm{Aut}\,\Delta$ one of the inclusions $C \subseteq Cg$, $C \subseteq C^*g$, $C^* \subseteq Cg$, $C^* \subseteq C^*g$ holds.*

Proof. See [2]. ∎

Since Ω_x has more than one infinite connected component, the graph Γ of 1.2 has more than one end. To show that Γ is standard, we restrict the possibilities for the cut C of 2.1. The following argument recurs, so we formalise it. For the rest of the paper, the cut C is assumed to satisfy 2.1.

2.2 Lemma. *Let x, y, z, u, v be vertices of Γ with $x, y \in C$ and $z, u, v \in C^*$. Then there is no $g \in (\mathrm{Aut}\,\Gamma)_x$ with $yg = z$ and $ug = v$.*

Proof. The number of points at distance $d(x, y)$ from x is finite, so as g maps y to z it must map some point of C^* into C. But then none of the inclusions of 2.1 holds. ∎

In the rest of the paper, Π is the branch on some infinite two-way line ℓ, and x is a vertex of ℓ. By 1.3, there is a positive integer $h > d+1$ such that $\Pi \cap \{w \in V\Gamma : d(x, w) > h\}$ has two components B_1 and B_2 and one of the following holds:

(a) $B_1 \subseteq C$, $\quad B_2 \subseteq C^*$

(b) $B_1 \subseteq C^*$, $\quad B_2 \subseteq C$

(c) $B_1, B_2 \subseteq C$

(d) $B_1, B_2 \subseteq C^*$.

2.3 Lemma. *In the above notation, if (c) holds, then $\Pi \subseteq C$. (By symmetry, if (d) holds, then $\Pi \subseteq C^*$.)*

Proof. There are three stages.

1. Claim. *Suppose that* $B_1, B_2 \subseteq C$ *and* $u, v \in \Pi \cap C^*$. *Then* u, v *lie on the same* Π *cross-section.*

Proof of 1. Suppose not. Let $z \in B_1$ and assume without loss that v is chosen in $\Pi \cap C^*$ so that $d(z, v)$ is maximal; we also may suppose that $v \in \ell$. First, it is clear that all geodesics from u to v lie in C^*. For if not, then choose $t \in \ell$ with $d(z, t) > d(z, v)$, $t \notin B_1$ and $d(v, t) = d(v, u)$. There is $g \in (\mathrm{Aut}\,\Gamma)_v$ with $ug = t$, and as the points in C on the (u, v) geodesic are mapped to other points in C, lemma 2.2 is contradicted.

Second, we may suppose that $u \not\sim v$. For if $u \sim v$, choose a vertex $w \in B_2 \cap \ell$ with $d(v, w) = d(v, z)$. There is $g \in (\mathrm{Aut}\,\Gamma)_v$ with $zg = w$, contradicting 2.2 as $ug \in C$.

Hence there is a geodesic from u to v lying in C^* and containing some points not on Π. It is easy to see that there are two vertices r and s on this geodesic with $r \sim s$, $d(z, s) = d(z, v)$ and some geodesic from z to s passing through r. There is $g \in (\mathrm{Aut}\,\Gamma)_z$ with $sg = v$. By 2.2, the vertex rg lies in C^*. But now, since rg and v are adjacent vertices, both lying in C^* but in different cross-sections of Π, the remark in the last paragraph applies.

2. Claim. *We suppose that* $B_1, B_2 \subseteq C$, *but that* x *and possibly some other points on the* Π *cross-section of* x *lie in* C^*. *As* C *is non-trivial, there is an infinite one-way branch* χ *contained in* C^*. *(For since* δC *is finite, there must be a branch which, sufficiently far along, lies entirely in* C^*.*) Let* $z \in B_1$ *with* $d(x, z)$ *sufficiently large, and let* $w \in \chi$ *with* $d(x, w) = d(x, z)$. *Then no geodesic from* x *to* w *passes through any point of* C.

Proof of 2. Suppose instead that some $u \in C$ lies on an (x, w) geodesic. There is $g \in (\mathrm{Aut}\,\Gamma)_x$ with $wg = z$. By 1 above, $ug \in C$, contradicting 2.2.

3. Claim. *The notation is that of 2 above, except that all* (x, w) *geodesics lie in* C^*. *This set of conditions cannot be satisfied.*

Proof of 3. Let $y, t \in \ell$ with $y, t \sim x$ (so $d(y, t) = 2$) and $d(z, y) = d(z, x) - 1$. There are two cases.

(a) $d(w, y) = d(w, x)$. By distance transitivity, there is $g \in (\mathrm{Aut}\,\Gamma)_w$ with $xg = y$. By 2.2, the image Πg must be an infinite two-way branch passing through y and lying in C^* (except possibly for some vertices lying on the Πg cross-section of y). Choose $v \in \ell g$ with $d(y, v) = 2$. Now $d(y, v) = d(y, t)$, so there is $h \in (\mathrm{Aut}\,\Gamma)_y$ with $th = v$. By part 1, $xh \in C^*$, but this contradicts 2.2.

(b) $d(w, y) = d(w, x) + 1$. Again we subdivide.

(b1) *The point adjacent to* y *in each* (y, w) *geodesic lies in* C^*.

Choose $s \in B_2$ with $d(y, s) = d(y, w) = d(x, s) + 1$. There is $g \in (\mathrm{Aut}\,\Gamma)_y$ with $sg = w$. By assumption, $xg \in C^*$, so 2.2 is contradicted.

(b2) *Some* (y, w) *geodesic passes through a point* u *in* C *adjacent to* y.

There is $g \in (\mathrm{Aut}\,\Gamma)_w$ with $xg = u$. As in (a), Πg is an infinite two-way branch contained in C^* (except possibly for some points on the Πg cross-section of u). There are again two cases.

(b2.1) $u \not\sim x$. Now choose a vertex v on ℓg with $v \sim yg$ and $d(u, v)=2$; so ℓg contains a (u, v) geodesic. Since $d(u, v)=d(u, x)$, there is $h\in(\mathrm{Aut}\,\Gamma)_u$ with $xh=v$. Now $yh\in C^*$, contradicting 2.2.

(b2.2) $u \sim x$. Let v be as in (b2.1). Now $u \sim t$; for if not, then $d(u, t)=d(u, v)$, so we have $g_1\in(\mathrm{Aut}\,\Gamma)_u$ with $tg_1=v$, again contradicting 2.2. Since the vertex u is adjacent to each of y, x, t, it lies on the same Π cross-section as x, so $d(z, u)=d(z, x)$. Let $yg=y'$, $tg=t'$ and $zg=z'$, so $v \sim y'$. By arguments similar to those above, $x \sim t'$ and $x \sim y'$, so x lies on the same Πg cross-section as u, and $d(z', x)=d(z', u)$.

Let $S=\{a\in C\colon a \sim u \text{ and } a \sim x\}$, $T=\{a\in C^*\colon a \sim u \text{ and } a \sim x\}$.

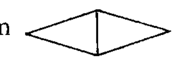
Clearly $y\in S$ and $y'\in T$. There is some $h_1\in(\mathrm{Aut}\,\Gamma)_z$ with $uh_1=x$. Let s be a vertex in T. Then either $sh_1=u$ or $sh_1\in S$. For $sh_1\in C$ (otherwise 2.2 is contradicted) and $sh_1 \sim x$; also, $sh_1=u$ or $sh_1 \sim u$, since otherwise we map (u, sh_1) to (u, v) and contradict 2.2. Hence $Th_1\subseteq S\cup\{u\}$. If there is $s\in T$ with $sh_1=u$, then $Th_1\subseteq(S\setminus\{xh_1\})\cup\{u\}$, where clearly $xh_1\in S$; if on the other hand $sh_1 \sim u$ for every $s\in T$, then $Th_1\subseteq S$. Since $h_1^{-1}y\notin T$, it follows in either case that $|T|<|S|$. Further, by considering some $h_2\in(\mathrm{Aut}\,\Gamma)_{z'}$ with $xh_2=u$, we can show similarly that $|T|>|S|$. This is a contradiction. ∎

2.4 Lemma. *Suppose that $C\cap\Pi$ and $C^*\cap\Pi$ are non-empty. Then there is a cross-section T of Π with $T\subseteq C$, and $\Pi\setminus T$ has two components L and R, both infinite, with $L\subseteq C$ and $R\subseteq C^*$.*

Proof. First we show that if S is a cross-section of Π, then $S\subseteq C$ or $S\subseteq C^*$. Suppose instead that $u, v\in S$ with $u\in C$ and $v\in C^*$. Let the components of $\Pi\setminus S$ be L_1 and R_1, and let h (where $h>d+1$) be a positive integer such that $R_1\cap\{z\in V\Gamma\colon d(u, z)>h\}\subseteq C$. Choose $x\in R_1$ with $d(x, u)$ sufficiently large that no geodesic from a vertex of S to a vertex on an edge of δC passes through x. By 2.3, there is a positive integer k such that $L_1\cap\{z\in V\Gamma\colon d(u, z)>k\}\subseteq C^*$. By distance transitivity, there is $g\in(\mathrm{Aut}\,\Gamma)_x$ with $ug=v$. It is clear from 2.2 that the infinite one-way branch L_1g lies entirely in C. By the choice of x, $(R_1\cap\{z\in V\Gamma\colon d(u, z)\geqq d(u, x)\})g\subseteq C$. Hence, by 2.3, $ug=v\in C$, which is a contradiction.

There is a further case to eliminate. Suppose that a Π cross-section T is contained in C and that $\Pi\setminus T$ has two components L and R with $R\subseteq C^*$. Let t be a point in T. For a contradiction, assume that L has a Π cross-section S in C^*. By 2.3, there is some positive integer h such that $L\cap\{z\in V\Gamma\colon d(t, z)>h\}\subseteq C$. Choose $u, u'\in\ell\cap R$ with $u \sim u'$ and $d(t, u)=d(t, u')+1$; also $v, v'\in\ell\cap L\cap\{z\in V\Gamma\colon d(t, z)>h\}$, with $v \sim v'$ and $d(t, v)=d(t, v')-1$. There is $g\in\mathrm{Aut}\,\Gamma$ with $(u, v)g=(u', v')$. This contradicts the assumption that C satisfies 2.1. ∎

3. Analysis of the cycles in Γ.

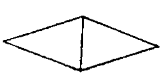
As mentioned in section 1, if Γ is a non-standard infinite distance transitive graph of finite valency, then Γ has non-degenerate cycles of length greater than three, or subgraphs of the form . To complete the proof of 1.2, we apply section 2 to show that this cannot happen.

Case (i). The graph Γ has a non-degenerate even cycle of length $2h$.

Let Π, ℓ be as usual, and choose $x, y \in \ell$ with $d(x, y) = h$. By distance transitivity, x and y lie on a cycle of length $2h$. Choose z, w on this cycle with $z \sim x$, $w \sim y$, $z \in \ell$ and $d(z, w) = h$. Let the Π cross-section of x be T, and let $\Pi \setminus T$ have components L and R with $y \in R$. By distance transitivity and 2.4, we may suppose that the cut C of 2.1 satisfies $R \subseteq C$ and $L \cup T \subseteq C^*$. Now C contains z and w but not all (z, w) geodesics. There is $g \in \mathrm{Aut}\,\Gamma$ with $(z, w)g = (x, y)$. The cut Cg contains x and y but not all (x, y) geodesics. This contradicts 2.4.

Case (ii). There is a non-degenerate odd cycle of length $2h+1 \geqq 5$ in Γ.

With Π and ℓ as usual, choose $x, y \in \ell$ with $d(x, y) = h$. The vertices x and y lie on a non-degenerate odd cycle of length $2h+1$. Let u and v lie on this cycle with $u \sim x$, $v \sim y$ and $u, v \notin \ell$. (As $2h+1 \geqq 5$, u and v are distinct.) Let T_x and T_y be the Π cross-sections containing x and y respectively; the components of $\Pi \setminus T_x$ are L_x and R_x with $y \in R_x$, and the components of $\Pi \setminus T_y$ are L_y and R_y with $x \in L_y$. We may suppose that the cut C of 2.1 satisfies $C \supseteqq L_x \cup T_x$, and $C^* \supseteqq R_x$. Now $u \in C$. For suppose that $u \in C^*$. Let $w, z \in \ell$ with $w, z \sim x$ and $d(y, z) = h-1$. There is $g \in (\mathrm{Aut}\,\Gamma)_z$ with $ug = w$, contradicting 2.2. Further, v and all (x, v) geodesics through u lie in C, since otherwise we could contradict 2.2 by mapping (x, v) to (x, y). By distance transitivity, there is $h \in \mathrm{Aut}\,\Gamma$ with $(x, y)h = (y, x)$. By 2.4, it is easy to see that $Ch \supseteqq R_y \cup T_y$ and $(Ch)^* \supseteqq L_y$. As before, u and v and all (u, v) geodesics lie in Ch. This contradicts 2.1.

Case (iii). The graph Γ has a subgraph of the form 

By distance transitivity, we may suppose that there are $x, y, z \in \ell$ with $y \sim x, z$ and $x \nsim z$, and also $w \in \Pi$ with $w \sim x, y, z$. Let T be the Π cross-section of y and w, and let $\Pi \setminus T$ have components L and R with $x \in L$ and $z \in R$. Without loss the cut C of 2.1 satisfies $C \supseteqq R$ and $C^* \supseteqq L \cup T$. There is some $g \in (\mathrm{Aut}\,\Gamma)_w$ with $zg = y$. The cut Cg contains y but not w, which contradicts 2.4, as y and w lie on the same Π cross-section.

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